Different Approaches for Bosonization in Higher Dimensions

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Abstract

We describe two distinct approaches for bosonization in higher dimensions; one is based on a direct comparison of current correlation functions while the other relies on a Master lagrangean formalism. These are used to bosonize the Massive Thirring Model in three and four dimensions in the weak coupling regime but with an arbitrary fermion mass. In both approaches the explicit bosonized lagrangean and current are derived in terms of gauge fields. The complete equivalence of the two bosonization methods is established. Exact results for the free massive fermion theory are also obtained. Finally, the two-dimensional theory is revisited and the possibility of extending this analysis for arbitrary dimensions is indicated.

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1) Introduction

Bosonization is a very powerful method by which fermionic theories are mapped into bosonic ones. It was first created and fully developed in the realm of two-dimensional physics. For some time it was thought that bosonization was only possible because of the strict constraint imposed by the dimensionality of space. Further research, however, revealed a deep structure underlying the process of bosonization. In particular, it became clear that a fundamental feature of this process is the fact that the basic fermions are mapped into the topological excitations of the associated bosonic theory. Correspondingly, the fermionic current is mapped into the topological current in the bosonic version and therefore its conservation is automatically implied. More recently, it has been found that this structure was not restricted to any specific dimension. The idea of bosonization was then generalized to higher dimensions by different methods [1, 2, 3, 4, 5, 6, 7, 8].

One of these methods which has been useful in studying abelian bosonization is the so called Master Lagrangean formalism [2, 3, 4, 7]. This is based on the coupling of the fermion to a dynamical boson gauge field which is a vector in three dimensions, a second rank antisymmetric tensor in four dimensions and so on. Integration over the bosonic degrees of freedom leads to the original fermionic theory, whereas integration over the fermion field yields the corresponding bosonized theory. Another approach which has been exploited even for the nonabelian bosonization is the hamiltonian constrained formulation [5]. This consists in embedding techniques which allow to convert the original fermionic theory into its bosonized form. An outline of a third method [8] was recently introduced which is based on the direct comparison of current correlation functions. This will be fully developed and exploited in the present work. In spite of the variety of bosonization methods in dimensions higher than two, a direct operator bosonization of the fermion field which leads to a complete description of the fermion correlators, as well as the expressions for the lagrangian and current operators in the framework of the associated bosonic theory was only obtained [1] in the case of a free massless fermion in 2+1D. On the other hand, the connection among the

different bosonization methods has not been established. Among other points which could be stressed, one is the fact that the bosonization of an interacting theory for an arbitrary fermion mass was still missing, despite the number of available techniques. Another one is the lack of any discussion on the apparent nonrenormalizability of the fermionic theory in contradistinction to its bosonic counterpart.

In this work, we address some of the important issues raised above. The method of current correlators is applied to the bosonization of the Massive Thirring Model (MTM) both in three and four dimensions in the weak coupling regime but with an arbitrary fermion mass. Explicit forms for the bosonized current and lagrangian are obtained in terms of a generalized free vector gauge field. From the general structure of these expressions one is able to conclude that similar results hold in arbitrary dimensions. The issue of the nonrenormalizability of the MTM and its implications for the bosonization are analyzed. We then reconsider the bosonization of the MTM by means of the Master Lagrangean approach. Again, the explicit expressions for the bosonized current and lagrangean both in three and four dimensions are derived but now these are given, respectively, in terms of a vector gauge field and of a second rank antisymmetric tensor gauge field. In both cases they are also generalized free gauge fields in the weak coupling regime of the MTM. This clearly shows the difference from the approach based on the comparison of current correlators where the bosonic field is always a vector. Nevertheless, we shall explicitly demonstrate the complete equivalence between the two bosonization schemes. In particular, the mapping connecting the bosonic gauge fields in the two methods is obtained. At the end, we revisit the bosonization of two-dimensional Massless Thirring Model, through the application of both the above mentioned methods, showing how the well known results are easily reproduced.

2) Massive Thirring Model in Three Dimensions

In this section we investigate the bosonization of the MTM by first developing the recently introduced approach based on the direct comparison of current correlation

functions [8] and then by the so called Master Lagrangean approach [2, 3, 7]. We also explicitly show the equivalence between the two methods.

2.1) Current Correlator Approach

Our starting point is the current correlators generating functional in the euclidean space

$$Z[J] = \int D\psi D\bar{\psi} \exp \left\{ - \int d^3z \left[\bar{\psi}(-\not{\partial} + m)\psi - \frac{\lambda^2}{2} j_\mu j_\mu + i\lambda j_\mu J_\mu \right] \right\} \tag{2.1}$$

where $j^\mu = \bar{\psi}\gamma^\mu\psi$ and we follow the metric of [9].

The current-current interaction can be described in terms of an auxiliary vector field in the following way

$$Z[J] = \int D\psi D\bar{\psi} DA_\mu \exp \left\{ - \int d^3z \left[\bar{\psi}(-\not{\partial} + m + i\lambda \not{A})\psi - \frac{1}{2} A_\mu A_\mu + i\lambda j_\mu J_\mu \right] \right\} \tag{2.2}$$

In the weak coupling approximation, only the two-legs one-loop graph contributes to the fermion determinant so that

$$Z[J] = \int DA_\mu \exp \left\{ - \int d^3z \left\{ \frac{\lambda^2}{2}(A_\mu + J_\mu)\Pi_{\mu\nu}(A_\nu + J_\nu) - \frac{1}{2} A_\mu A_\mu \right\} \right\} \tag{2.3}$$

The one-loop vacuum polarization tensor $\Pi_{\mu\nu}$ has been computed both by using lattice [9] and continuum [10] regularizations, giving in momentum space

$$\Pi_{\mu\nu}(q) = A(q^2)C_{\mu\nu}(q) + B(q^2)P_{\mu\nu}(q) \tag{2.4}$$

where $C_{\mu\nu}(q) = \epsilon_{\mu\nu\alpha}q_\alpha$ and $P_{\mu\nu}(q) = q^2\delta_{\mu\nu} - q_\mu q_\nu$, while

$$A(q^2) = a_0 + \frac{1}{4\pi}\int_0^1 dt\{1 - m[m^2 + t(1-t)q^2]^{-1/2}\} \tag{2.5}$$

and

$$B(q^2) = \frac{1}{2\pi}\int_0^1 dtt(1-t)[m^2 + t(1-t)q^2]^{-1/2} \tag{2.6}$$

Note that the parity violating term, as is well known, contains a regularization dependent finite term proportional to a_0 [9, 10].

The following algebraic relations among $C_{\mu\nu}$ and $P_{\mu\nu}$ will prove to be very useful in what follows:

$$C_{\mu\alpha}C_{\alpha\nu} = -P_{\mu\nu} \quad ; \quad P_{\mu\alpha}P_{\alpha\nu} = q^2 P_{\mu\nu} \quad ; \quad C_{\mu\alpha}P_{\alpha\nu} = P_{\mu\alpha}C_{\alpha\nu} = q^2 C_{\mu\nu} \tag{2.7}$$

After integrating over A_μ in (2.3), we obtain

$$Z[J] = \exp\left\{ -\int d^3z \frac{\lambda^2}{2} \left\{ J_\mu \Pi_{\mu\nu} J_\nu + \lambda^2 J_\lambda \Pi_{\lambda\mu} \Gamma_{\mu\nu} \Pi_{\rho\nu} J_\rho \right\} \right\} \tag{2.8}$$

where $\Gamma_{\mu\nu} = [\delta_{\mu\nu} - \lambda^2 \Pi_{\mu\nu}]^{-1} = \delta_{\mu\nu} + O(\lambda^2)$. Since we are working up to the second order in λ, therefore, we can just retain the $\delta_{\mu\nu}$ piece in (2.8).

The two-point current correlation function is easily obtained from (2.8)

$$-\frac{1}{\lambda^2}\frac{\delta^2}{\delta J_\mu(q)\delta J_\nu(-q)}|_{J=0} =< j_\mu(q) j_\nu(-q) >= \Pi_{\mu\nu}(q) + \lambda^2 \Pi_{\mu\alpha}(q)\Pi_{\alpha\nu}(q) \tag{2.9}$$

Now, it is not difficult to extract the bosonized lagrangean of the MTM in this limit. This is given by

$$\mathcal{L}_{MTM} = \frac{1}{2} B_\mu \left(\Pi_{\mu\nu} + \lambda^2 \Pi_{\mu\alpha}\Pi_{\alpha\nu} \right) B_\nu \tag{2.10}$$

Observe that this is a gauge theory, because of the transverse nature of the kernel. It is nonlocal for any value of the fermion mass except for $m \to \infty$ as can be explicitly checked from expressions (2.5) and (2.6). This is a general feature of bosonization in higher dimensions [1, 6, 7, 8] occurring even in the free case. It has been shown that generalized free (quadratic) gauge theories of this type, in spite of being nonlocal, yield sensible results [6, 7, 11]. In particular, they respect causality.

It is now easy to deduce the current bosonization formula which will reproduce the correlation function (2.9). This is given by

$$j_\mu = \left(\Pi_{\mu\nu} + \lambda^2 \Pi_{\mu\alpha}\Pi_{\alpha\nu} \right) B_\nu \tag{2.11}$$

To see that this is correct, we must consider the two-point correlation funtion of the B_μ-field which is given by the inverse of the kernel appearing in the quadratic lagrangean (2.10). In order to perform this inversion we add a gauge fixing term $\xi B(q^2) q_\mu q_\nu$. As it turns out it will also be convenient to add the same longitudinal

term to the second $\Pi_{\mu\nu}$ in the λ-dependent term of (2.10) which is allowed by the transversality of $\Pi_{\mu\nu}$. The result for the field two-point function in this gauge is

$$< B_\mu(q) B_\nu(-q) >= \left[\Pi_{\mu\alpha}(q)\left(\delta_{\alpha\nu} + \lambda^2\left(\Pi_{\alpha\nu}(q) + \xi B(q^2) q_\alpha q_\nu\right)\right) + \xi B(q^2) q_\mu q_\nu\right]^{-1}$$

$$= D_{\mu\nu}(q) - \lambda^2 \frac{P_{\mu\nu}(q)}{q^2} + O(\lambda^4) \tag{2.12}$$

where

$$D_{\mu\nu}(q) = \left[\Pi_{\mu\nu}(q) + \xi B(q^2) q_\mu q_\nu\right]^{-1}$$

$$= \frac{1}{q^2[A^2(q^2) + q^2 B^2(q^2)]}\left[B(q^2) P_{\mu\nu} - A(q^2) C_{\mu\nu}\right] + \frac{1}{\xi}\frac{q_\mu q_\nu}{q^4 B(q^2)} \tag{2.13}$$

Now, using (2.11), we have

$$< j_\mu(q) j_\nu(-q) >= \left(\Pi_{\mu\alpha} + \lambda^2 \Pi_{\mu\beta}\Pi_{\beta\alpha}\right)(q)\left(\Pi_{\nu\rho} + \lambda^2 \Pi_{\nu\sigma}\Pi_{\sigma\rho}\right)(-q) < B_\alpha(q) B_\rho(-q) > \tag{2.14}$$

It is straightforward to see that the current correlation function (2.9) is reproduced by inserting (2.12) in (2.14). Also, it is not difficult to prove that all higher correlation functions are reproduced by the bosonic expression (2.11). Note that, in particular, the odd functions vanish because the odd B_μ-correlators are zero in the quadratic gauge field theory (2.10). The current bosonization formula (2.11) is thereby confirmed, also showing that in this limit the MTM is mapped into a generalized free gauge theory (2.10).

Similarly to the two-dimensional case, we can now define a dual current as

$$\bar{j}_\mu(q) = \frac{i}{q} C_{\mu\alpha}(q) j_\alpha(q) \tag{2.15}$$

The correlation functions of this dual current are identical to those of j_μ as can be be verified from the bosonized expression (2.11). This generalizes a similar duality relation found in the large mass limit [2].

Let us next consider the exact bosonization of the free massive fermionic theory which is obtained in the limit when the Thirring coupling λ vanishes. From (2.10) and (2.11) we immediately obtain the bosonization formulae for the lagrangean and current, namely

$$\bar{\psi}(-i \not{\partial} + m)\psi|_{\text{free}} = \frac{1}{2} B_\mu \Pi_{\mu\nu} B_\nu \tag{2.16}$$

$$j_\mu(q)|_{\rm free} = \Pi_{\mu\nu} B_\nu \tag{2.17}$$

For a vanishing mass, these expressions reduce exactly to the ones found by following a direct operator bosonization of the free massless fermion field [1]. This clearly shows that the operator realization obtained in [1] is exact and no nonquadratic corrections are necessary.

A very important point must be stressed now. These exact bosonization formulae are strictly valid only in the free case. In an interacting theory, both expressions are modified in general. The current, for instance, is given by (2.11), while the kinetic fermion lagrangian for the MTM can be obtained from (2.10) and (2.11) giving the result

$$\bar\psi(-i\not\partial + m)\psi|_{\rm MTM} = \bar\psi(-i\not\partial + m)\psi|_{\rm free} + \frac{\lambda^2}{2} B_\mu(q)\Big(\Pi_{\mu\alpha}\Pi_{\alpha\nu}(q) + \Pi_{\mu\alpha}\Pi_{\alpha\nu}(-q)\Big)B_\nu(-q) \tag{2.18}$$

This relation clearly shows the modification of the bosonization formula for the kinetic fermion lagrangian which is produced when we add the Thirring interaction to the free massive lagrangian. Self- consistency is preserved as can be inferred by setting $\lambda = 0$ in the above formula. The fact that the bosonization formulae are changed in the presence of an interaction is a general feature of dimensions higher than two [7].

Our results demonstrate that, because of the finiteness of the one-loop vacuum polarization tensor, even though the MTM is pertubatively nonrenormalizable by the usual power counting criterion, at least in the weak coupling regime, it yields sensible results.

2.2) Master Lagrangean Approach

This method starts by considering the theory of a fermion interacting with a vector gauge field which, by its turn, has a topological interaction with a dynamical Maxwell field [2]. The corresponding lagrangean is called Master Lagrangean since integrating out the gauge fields leads to the MTM, while the bosonized form is obtained by integrating over the fermion field. It is given by [2]

$$\mathcal{L}_M = \bar\psi(-\not\partial + m + i\lambda \not f)\psi - \frac{1}{4}F_{\mu\nu}F_{\mu\nu} - i\epsilon_{\mu\nu\lambda} f_\mu \partial_\nu A_\lambda \tag{2.19}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The corresponding current correlators generating functional in euclidean space is given by

$$Z[J] = \int D\psi D\bar{\psi} Df_\mu DA_\mu \delta(\partial_\mu f_\mu)\delta(\partial_\mu A_\mu) \exp\left\{-\int d^3z \left[\mathcal{L}_M + i\epsilon_{\mu\nu\lambda} J_\mu \partial_\nu A_\lambda\right]\right\} \tag{2.20}$$

where a covariant gauge fixing has been chosen for both the fields. It is easily seen by integrating out the gauge fields that the MTM current generating functional (2.1) is reproduced [2]. From this observation we can immediately conclude that the exact bosonization formula in momentum space for the Thirring current in terms of the A_μ-field is given by

$$j_\mu = -\frac{i}{\lambda} C_{\mu\nu} A_\nu \tag{2.21}$$

It is important to note at this point that the above result is exact and in particular does not depend on any weak coupling approximation in the Thirring coupling. Moreover, this is the only bosonization formula in dimensions higher than two which does not depend on the type of interacting theory in which the bosonized operator is embedded. We can see, for instance, that a completely different result is obtained through the current correlators method where both the lagrangean and current are modified as can be seen from (2.10) and (2.11). Let us stress that the vector gauge fields B_μ and A_μ which appear in the two bosonization formulas (2.11) and (2.21) of course are not the same. Later on, we will explicitly show the equivalence of the two descriptions and find out the relationship between these fields.

In order to obtain the bosonized form of the MTM lagrangean, let us integrate over the fermion fields. This will be done as before, by means of a weak coupling expansion. We find the generating functional

$$\begin{aligned} Z[J] = \int Df_\mu DA_\mu \delta(\partial_\mu f_\mu)\delta(\partial_\mu A_\mu) \exp\Big\{-\int d^3z \Big[-\frac{1}{4} F_{\mu\nu} F_{\mu\nu} \\ +\frac{\lambda^2}{2} f_\mu \Pi_{\mu\nu} f_\nu - i\epsilon_{\mu\nu\lambda}(f_\mu - J_\mu)\partial_\nu A_\lambda\Big]\Big\} \end{aligned} \tag{2.22}$$

where $\Pi_{\mu\nu}$ is the one-loop vacuum polarization tensor given in momentum space by (2.4). Now, performing the quadratic functional integration over f_μ we get

$$Z[J] = \int DA_\mu \delta(\partial_\mu A_\mu) \exp\left\{-\int d^3z \left[\frac{1}{2} A_\mu \Sigma_{\mu\nu} A_\nu + i\epsilon_{\mu\nu\lambda} J_\mu \partial_\nu A_\lambda\right]\right\} \tag{2.23}$$

where

$$\Sigma_{\mu\nu} = \left(\frac{B(q^2)}{\lambda^2(A^2(q^2)+q^2B^2(q^2))} - 1 \right) P_{\mu\nu} - \frac{A(q^2)}{\lambda^2(A^2(q^2)+q^2B^2(q^2))} C_{\mu\nu} \qquad (2.24)$$

The complete bosonization of the MTM lagrangean in the weak coupling limit is therefore given by

$$\mathcal{L}_{MTM} = \frac{1}{2} A_\mu \Sigma_{\mu\nu} A_\nu \qquad (2.25)$$

which, by the transversality of $\Sigma_{\mu\nu}$, is also a generalized free gauge theory. Note that just as the bosonization of the current in terms of the A_μ-field (2.21) was shown to differ from that obtained by the previous method, the same happens for the corresponding bosonized lagrangeans. However, as we now show, the current correlation functions derived in the present formalism are identical to (2.9), which was also reproduced by the previous bosonization formula (2.11) in terms of the B_μ-field. Indeed, performing the A_μ-field integration in (2.23) and keeping terms up to the order $O(\lambda^2)$, we obtain

$$Z[J] = \exp\left\{ - \int d^3q \frac{\lambda^2}{2} J_\mu(q) \left[\Pi_{\mu\nu}(q) + \lambda^2 \Pi_{\mu\alpha}(q) \Pi_{\alpha\nu}(q) \right] J_\nu(-q) \right\} \qquad (2.26)$$

Now, it is straightforward to see that by taking functional derivatives of (2.26) with respect to the sources the two-point correlation function (2.9) is reproduced. Similarly, all the higher current correlators can be obtained as well. As usual, the odd ones will vanish.

We have shown how the two distinct methods of bosonization considered here can give equivalent descriptions of the same fermionic theory. It is also possible to provide a direct mapping between the basic bosonic fields in the different approaches, namely, the A_μ and B_μ fields. Any divergenceless field in three dimensions like A_μ in the covariant gauge we are working can be expressed as

$$A_\mu = \left[a(q^2) P_{\mu\alpha} + b(q^2) C_{\mu\alpha} \right] B_\alpha \qquad (2.27)$$

where $a(q^2)$ and $b(q^2)$ are scalar functions and B_μ is some other vector field. Identifying A_μ and B_μ, respectively, with our two bosonic gauge fields and inserting (2.27) in

(2.21), we can determine the unknown functions a and b by comparing the resulting expression with (2.11). We find

$$a(q^2) = i\lambda \left[\frac{A(q^2)}{q^2} + 2\lambda^2 A(q^2) B(q^2) \right]$$

$$b(q^2) = i\lambda \left[-B(q^2) + \lambda^2 (A^2(q^2) - q^2 B^2(q^2)) \right] \tag{2.28}$$

It can now be verified that inserting (2.27) with (2.28) in the bosonic lagrangean of the MTM, eq. (2.25), we precisely reobtain the alternative form given in (2.10). This establishes the complete equivalence of the two bosonization methods.

We now discuss the inverse of the mapping expressed by (2.27). The operator multiplying B_μ in (2.27) is not invertible because of transversality. Hence, taking advantage of the gauge condition satisfied by B_μ, we can add an extra longitudinal piece as we did in (2.12), to obtain

$$A_\mu = \left[a(q^2) P_{\mu\alpha} + b(q^2) C_{\mu\alpha} + \xi B(q^2) q_\mu q_\alpha \right] B_\alpha \tag{2.29}$$

We can now invert the relevant operator and get

$$B_\mu = \frac{1}{q^2 (b^2(q^2) + q^2 a^2(q^2))} \left[a(q^2) P_{\mu\alpha} - b(q^2) C_{\mu\alpha} \right] A_\alpha \tag{2.30}$$

where explicit use has been made of the covariant gauge condition satisfied by A_μ. It can be shown that using (2.30), in the bosonized expressions for the lagrangian and current given respectively by (2.10) and (2.11), we reproduce the corresponding ones in terms of A_μ, namely (2.25) and (2.21). This establishes the algebraic consistency of the whole program.

We conclude the section by giving the exact explicit expressions for the lagrangean and the current for a free massive fermion, which follow from (2.25) and (2.21), respectively, by making the scaling $A_\mu \to \lambda A_\mu$ and then setting $\lambda = 0$, namely

$$\bar{\psi}(-i \not{\partial} + m)\psi|_{\text{free}} = \frac{1}{2} A_\mu \Sigma^f_{\mu\nu} A_\nu \tag{2.31}$$

$$j_\mu(q)|_{\text{free}} = -i C_{\mu\nu} A_\nu \tag{2.32}$$

where

$$\Sigma^f_{\mu\nu} = \frac{B(q^2)}{A^2(q^2)+q^2B^2(q^2)}P_{\mu\nu} - \frac{A(q^2)}{A^2(q^2)+q^2B^2(q^2)}C_{\mu\nu} \tag{2.33}$$

Note that the above equations are the analogs of (2.16) and (2.17) obtained in the current correlator approach. Observe also that the current has the general topological structure as (2.21).

As in the interacting theory, the fields A_μ and B_μ are related by an expression identical to (2.27) except for the fact that the scalar functions (2.28) are replaced by

$$a^f(q^2) = i\frac{A(q^2)}{q^2}$$

$$b^f(q^2) = -iB(q^2) \tag{2.34}$$

Just as in the previous approach, the results (2.31) and (2.32) are strictly exact only in the free theory.

3) Massive Thirring Model in Four Dimensions

Let us now consider the bosonization of the MTM in four dimensions by following the two methods applied in the previous section. An important difference from the three-dimensional case is that now the one-loop vacuum polarization tensor is no longer finite and renormalization is necessary. The implications of this will be analyzed.

3.1) Current Correlator Approach

Again we start from the current correlators generating functional in four dimensional euclidean space

$$Z[J] = \int D\psi D\bar{\psi} \exp\left\{-\int d^4z \left[\bar{\psi}(-\not\partial + m)\psi - \frac{\lambda^2}{2}j_\mu j_\mu + i\lambda j_\mu J_\mu\right]\right\} \tag{3.1}$$

As usual this can be written as

$$Z[J] = \int D\psi D\bar{\psi} DA_\mu \exp\left\{-\int d^4z \left[\bar{\psi}(-\not\partial + m + ie\not A)\psi - \frac{\mu^2}{2}A_\mu A_\mu + i\lambda j_\mu J_\mu\right]\right\} \tag{3.2}$$

where $\lambda = \frac{e}{\mu}$. We now integrate over the fermion field in the small coupling approximation where only the two-legs one-loop graph does contribute. We obtain

$$Z[J] = \int DA_\mu \exp\left\{-\int d^4z \left\{\frac{e_R^2}{2}(A_\mu + \frac{J_\mu}{\mu})\Pi_{\mu\nu}(A_\nu + \frac{J_\nu)}{\mu} - \frac{\mu^2}{2}A_\mu A_\mu\right\}\right\} \tag{3.3}$$

where the vacuum polarization tensor is given by [12]

$$\Pi_{\mu\nu}(q) = (q^2\delta_{\mu\nu} - q_\mu q_\nu)\Pi(q^2) = P_{\mu\nu}\Pi(q^2)$$

with

$$\Pi(q^2) = -\frac{1}{12\pi^2}\left\{\frac{1}{3} + 2\left(1 - \frac{2m^2}{q^2}\right)\left[\frac{1}{2}x \ln\frac{x+1}{x-1} - 1\right]\right\} \tag{3.4}$$

in which $x = \left(1 + \frac{4m^2}{q^2}\right)^{1/2}$. In the above expression, the renormalized coupling constant e_R is given, in lowest order, by

$$e_R^2 = \left[1 - \frac{e_R^2}{12\pi^2}\ln\Lambda^2\right]e^2 \tag{3.5}$$

where Λ is an ultraviolet cutoff.

At this point, let us remark that the coupling constant e is renormalized exctly as in QED because the evaluation of the fermionic determinant in (3.2) was performed in the renormalizable sector of the effective theory defined by the action in (3.2). In this sense it is meaningful to use a small coupling expansion in e_R. On the other hand, by rescaling $A_\mu \to \mu A_\mu$, it is possible to redefine the coupling as $\frac{e_R}{\mu}$. Consistency then requires that an expansion in small e_R must be supplemented by the requirement that $e_R << \mu$. Note that the original MTM with the coupling $\lambda = \frac{e}{\mu}$ is nonrenormalizable. This discussion clarifies the precise meaning of a small coupling expansion in MTM: small coupling expansion in the effective theory implies the corresponding expansion in the MTM with a redefined coupling $\tilde{\lambda} = \frac{e_R}{\mu}$. For notational convenience we henceforth set $\mu = 1$, so that $\tilde{\lambda} = e_R$. Observe that in the three-dimensional case the one-loop vacuum polarization tensor was finite and therefore no renormalization was necessary. Hence there was no need to introduce a redefined Thirring coupling.

We can now perform the quadratic integration over A_μ in (3.3) by using the fact that up to the order $O(\tilde{\lambda}^2)$ the inverse of the kernel

$$\left[\delta_{\mu\nu} - \tilde{\lambda}^2\Pi_{\mu\nu}\right]^{-1} = \delta_{\mu\nu} + O(\tilde{\lambda}^2) \tag{3.6}$$

The result is

$$Z[J] = \exp\left\{ - \int d^4 z \frac{\tilde{\lambda}^2}{2} \left\{ J_\mu \Pi_{\mu\nu} J_\nu + \tilde{\lambda}^2 J_\lambda \Pi_{\lambda\mu} \Pi_{\rho\mu} J_\rho \right\} \right\} \tag{3.7}$$

Now, taking functional derivatives of the above expression with respect to the sources, we immediately find the two-point current correlation function

$$- \frac{1}{\tilde{\lambda}^2} \frac{\delta^2}{\delta J_\mu(q) \delta J_\nu(-q)} |_{J=0} = < j_\mu(q) j_\nu(-q) > = \Pi_{\mu\nu}(q) + \tilde{\lambda}^2 \Pi_{\mu\alpha}(q) \Pi_{\alpha\nu}(q) \tag{3.8}$$

The bosonized form of the MTM lagrangian in the small $\tilde{\lambda}$ regime is now easily inferred, namely

$$\begin{aligned} \mathcal{L}_{MTM} &= \frac{1}{2} B_\mu \left(\Pi_{\mu\nu} + \tilde{\lambda}^2 \Pi_{\mu\alpha} \Pi_{\alpha\nu} \right) B_\nu \\ &= \frac{1}{2} B_\mu \Pi(q^2) \left(1 + \tilde{\lambda}^2 q^2 \Pi(q^2) \right) P_{\mu\nu} B_\nu \end{aligned} \tag{3.9}$$

The simplification in the second line happens because the vacuum polarization tensor has only the $P_{\mu\nu}$ part in four dimensions. Note that this bosonized form of the lagrangian is structurally identical to the one obtained for the corresponding bosonization formula (2.10) in three dimensions. This also shows the practical viability of the current correlator approach for bosonization.

In the same way as in three dimensions, we now proceed to the bosonization of the current. This can be immediately obtained by inspecting the previous formula, given by (2.11), namely

$$\begin{aligned} j_\mu &= \left(\Pi_{\mu\nu} + \tilde{\lambda}^2 \Pi_{\mu\alpha} \Pi_{\alpha\nu} \right) B_\nu \\ &= \Pi(q^2) \left(1 + \tilde{\lambda}^2 q^2 \Pi(q^2) \right) P_{\mu\nu} B_\nu \end{aligned} \tag{3.10}$$

Following exactly the same steps as in the previous section, we can show that the above bosonized expression for j_μ precisely reproduces the two-point correlator (3.8), as well as the higher ones. The odd funtions as usual vanish. This confirms the validity of our bosonization procedure also in this case.

The free case can now be easily obtained as in the three-dimensional theory, by taking the limit $\tilde{\lambda} \to 0$. We obtain the exact identifications

$$\bar{\psi}(-i \not{\partial} + m)\psi|_{\text{free}} = \frac{1}{2} B_\mu \Pi_{\mu\nu} B_\nu = \frac{1}{2} B_\mu \Pi(q^2) P_{\mu\nu} B_\nu \tag{3.11}$$

and

$$j_\mu(q)|_{\rm free} = \Pi_{\mu\nu} B_\nu = \Pi(q^2) P_{\mu\nu} B_\nu \tag{3.12}$$

Let us remind again that in the presence of interaction both the above expressions are modified and, in particular, a relation exactly analogous to (2.18) is obtained.

3.2) Master Lagrangean Approach

Let us now apply the Master Lagrangean method to the MTM in four dimensions. In this case, we have to consider the theory of a fermion coupled to a vector gauge field which topologically interacts with a dynamical Kalb-Ramond second rank antisymmetric tensor gauge field [3, 7].

$$\mathcal{L}_M = \bar\psi(-\not\partial + m + i\lambda \not f)\psi - \frac{1}{3}F_{\mu\nu\lambda}F_{\mu\nu\lambda} - i\epsilon_{\mu\nu\alpha\beta} f_\mu \partial_\nu A_{\alpha\beta} \tag{3.13}$$

where $F_{\mu\nu\lambda} = \partial_\mu A_{\nu\lambda} - \partial_\nu A_{\mu\lambda} - \partial_\lambda A_{\nu\mu}$. The euclidean generating functional of current correlators associated with this is given by

$$Z[J] = \int D\psi D\bar\psi D f_\mu D A_{\mu\nu} D\alpha \delta(\partial_\mu f_\mu)\delta(\partial_\mu A_{\mu\nu} + \partial_\nu \alpha) \exp\left\{-\int d^4z \left[\mathcal{L}_M + i\epsilon_{\mu\nu\alpha\beta}\partial_\nu A_{\alpha\beta} J_\mu\right]\right\} \tag{3.14}$$

Notice that the delta-functional in the Kalb-Ramond field includes an additional term which is integrated on. This accounts for the reducibility in the usual covariant gauge fixing which is related to the fact that the gauge invariance in the Kalb-Ramond field is reducible. It can be easily verified that upon integration over the gauge fields we reproduce the MTM current generating functional (3.1). From this again we can immediately infer the exact bosonization formula for the current, namely

$$j_\mu = \frac{i}{\lambda}\epsilon_{\mu\nu\alpha\beta} q_\nu A_{\alpha\beta} \tag{3.15}$$

Note that in analogy with the three-dimensional result (2.21), the bosonized current is exactly given by the topological current in terms of the Kalb-Ramond field. The previous comments made below eq. (2.21) regarding the comparison between the two bosonization methods of the fermion current also apply here.

The bosonized form of the MTM lagrangean is now obtained from (3.14) by integrating out the fermion fields. The result in the weak coupling approximation is given by,

$$Z[J] = \int Df_\mu DA_{\mu\nu} D\alpha \delta(\partial_\mu f_\mu)\delta(\partial_\mu A_{\mu\nu} + \partial_\nu \alpha) \exp\Big\{-\int d^4z \Big[-\frac{1}{3}F_{\mu\nu\lambda}F_{\mu\nu\lambda}$$

$$-i\epsilon_{\mu\nu\alpha\beta} f_\mu \partial_\nu A_\alpha \beta + \frac{\tilde{\lambda}^2}{2} f_\mu \Pi_{\mu\nu} f_\nu + i\epsilon_{\mu\nu\alpha\beta}\partial_\nu A_{\alpha\beta} J_\mu\Big]\Big\} \tag{3.16}$$

where $\Pi_{\mu\nu}$ is the one-loop vacuum polarization tensor already given in momentum space by (3.4). Observe that, as explained before, the original MTM coupling λ is replaced by the redefined coupling $\tilde{\lambda}$. Next, performing the quadratic functional integration over f_μ we obtain

$$Z[J] = \int DA_{\mu\nu} D\alpha \delta(\partial_\mu A_{\mu\nu} + \partial_\nu \alpha) \exp\Big\{-\int d^4z \Big[\frac{1}{3}F_{\mu\nu\lambda}\Big[\frac{1}{q^2\tilde{\lambda}^2\Pi(q^2)} - 1\Big]F_{\mu\nu\lambda}$$

$$+i\epsilon_{\mu\nu\alpha\beta}\partial_\nu A_{\alpha\beta} J_\mu\Big]\Big\} \tag{3.17}$$

from which we can read the explict bosonization expression for the MTM lagrangean in the weak coupling limit and arbitrary mass, in terms of the second rank antisymmetric Kalb-Ramond field

$$\mathcal{L}_{MTM} = \frac{1}{3}F_{\mu\nu\lambda}\left[\frac{1}{q^2\tilde{\lambda}^2\Pi(q^2)} - 1\right]F_{\mu\nu\lambda} \tag{3.18}$$

Notice that in the weak coupling regime the first piece dominates leading to a positive definite lagrangean, as expected in a euclidean metric. The same also holds for the lagrangean (2.25) in the three dimensional case. Positive definiteness of the bosonised lagrangians obtained by the current correlator method is self evident, as is easily seen by looking at the respective expressions.

In the same way as has happened in the case of the current, notice that the bosonized lagrangian (3.18) also differs from the one obtained by the previous method, eq. (3.9). However, as we now explicitly show, the same fermionic current correlation functions (3.8) are reproduced. This can be easily seen by integrating over α and the Kalb-Ramond field in (3.17),

$$Z[J] = \exp\left\{-\int d^4z \frac{\tilde{\lambda}^2}{2} J_\mu \left[\Pi_{\mu\nu}(q) + \tilde{\lambda}^2 \Pi_{\mu\alpha}(q)\Pi_{\alpha\nu}(q)\right] J_\nu\right\} \tag{3.19}$$

and taking functional derivatives.

We shall now explicitly derive the relationship between the Kalb-Ramond field and the vector gauge field used in the current correlator approach, thereby generalizing the relation (2.27). Note that in the covariant gauge, any Kalb-Ramond field can always be written in terms of a vector field as

$$A_{\mu\nu} = g(q^2)\epsilon_{\mu\nu\alpha\beta}q_\alpha B_\beta \tag{3.20}$$

where $g(q^2)$ is some scalar function. By identifying the above gauge fields with those occurring in the two bosonization approaches cosidered above, it is possible to determine the scalar function by the direct comparison of the two bosonization formulas for the current (3.10) and (3.15). We find

$$g(q^2) = i\frac{\tilde{\lambda}}{2}\Pi(q^2)\left(1+\tilde{\lambda}^2 q^2\Pi(q^2)\right) \tag{3.21}$$

With this mapping, it is simple to check that the bosonized lagrangeans obtained in the two different approaches, namely (3.9) and (3.18) also become identical. Following the steps of the previous section we can invert relation (3.20) and thereby demonstrate the complete equivalence between the two approaches in both directions.

It is now simple to read off the exact results for the free massive theory by scaling $A_{\mu\nu} \to \tilde{\lambda}A_{\mu\nu}$ and finally making $\tilde{\lambda} = 0$:

$$\bar{\psi}(-i\not{\partial}+m)\psi|_{\mathrm{free}} = \frac{1}{3}F_{\mu\nu\lambda}\left[\frac{1}{q^2\Pi(q^2)}\right]F_{\mu\nu\lambda} \tag{3.22}$$

$$j_\mu(q)|_{\mathrm{free}} = i\epsilon_{\mu\nu\alpha\beta}q_\nu A_{\alpha\beta} \tag{3.23}$$

The relation mapping the two fields remains identical to (3.20), except for the fact that the scalar function $g(q^2)$ is modified as

$$g^f(q^2) = \frac{i}{2}\Pi(q^2) \tag{3.24}$$

4) The Two-Dimensional Theory Revisited

Let us consider here the application of the two methods discussed in the previous sections to the bosonization of the two-dimensional massless Thirring Model which is

known to be exactly solvable. The euclidean current correlator generating functional is given by

$$Z[J] = \int D\psi D\bar{\psi} \exp\left\{- \int d^2z \left[\bar{\psi}(-\not{\partial})\psi - \frac{e^2}{2} j_\mu j_\mu + ie\ j_\mu J_\mu\right]\right\} \tag{4.1}$$

As done previously we can eliminate the four-fermion interaction by introducing a vector field, namely

$$Z[J] = \int D\psi D\bar{\psi} DA_\mu \exp\left\{- \int d^2z \left[\bar{\psi}(-\not{\partial} + ie\ \not{A})\psi - \frac{1}{2} A_\mu A_\mu + ie\ j_\mu J_\mu\right]\right\} \tag{4.2}$$

The fermion integration can be done exactly giving the result

$$Z[J] = \int DA_\mu \delta(\partial_\mu A_\mu) \exp\left\{- \int d^2z \left[\frac{e^2}{2\pi}(A_\mu + J_\mu)\Gamma_{\mu\nu}(A_\nu + J_\nu) - \frac{1}{2} A_\mu A_\mu\right]\right\} \tag{4.3}$$

where

$$\Gamma_{\mu\nu} = \delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box} \tag{4.4}$$

Performing the gaussian integration in (4.3), we get

$$Z[J] = \exp\left\{- \int d^2z \left(\frac{e^2}{2(\pi - e^2)}\right) J_\mu \Gamma_{\mu\nu} J_\nu\right\} \tag{4.5}$$

It is now trivial to compute the current two-point correlation function by taking functional derivatives with respect to J_μ. The result is

$$-\frac{1}{e^2} \frac{\delta^2}{\delta J_\mu(x) \delta J_\nu(y)}|_{J=0} =< j_\mu(x) j_\nu(y) >= \left(\frac{1}{(\pi - e^2)}\right) \left[\delta_{\mu\nu}\delta^2(x - y) - \partial_\mu^x \partial_\nu^x \left[\frac{1}{\Box}\right](x - y)\right] \tag{4.6}$$

As done earlier, the bosonized form of the lagrangian is immediately inferred from this result,

$$\mathcal{L}_{Thirring} = \left(\frac{1}{2(\pi - e^2)}\right) B_\mu \Gamma_{\mu\nu} B_\nu \tag{4.7}$$

Similarly, the bosonization of the current yields

$$j_\mu = \left(\frac{1}{\pi - e^2}\right) \Gamma_{\mu\nu} B_\nu \tag{4.8}$$

Now it is simple to verify the validity of these bosonization rules by evaluating the current correlation function in the bosonic language. In order to do it, we obtain the

propagator of the B_μ-field from (4.7) in some covariant gauge and using it in (4.8), easily reproduce (4.6).

The results for the free theory follow by simply putting $e = 0$ in (4.7) and (4.8). This clearly shows why the Thirring model is a free theory since the free and interacting cases just differ by a normalization of the B_μ-field. Observe the distinction from the higher dimensional situation, where the connection between the free and interacting cases is highly nontrivial.

One may wonder how to relate this bosonization result to the usual one in which the bosonic field is scalar. This will be done by adopting the Master Lagrangean approach [3] where the corresponding lagrangean is now defined by

$$\mathcal{L}_M = \bar{\psi}(-\not\partial + ie \not A)\psi - \frac{1}{2}\partial_\mu\phi\partial_\mu\phi - i\epsilon_{\mu\nu}A_\mu\partial_\nu\phi \tag{4.9}$$

As before, the current correlators generating functional is given by

$$Z[J] = \int D\psi D\bar{\psi} D\phi DA_\mu \delta(\partial_\mu A_\mu) \exp\left\{-\int d^2z \left[\mathcal{L}_M + i\epsilon_{\mu\nu}\partial_\nu\phi J_\mu\right]\right\} \tag{4.10}$$

Doing the integration over the bosonic fields, one immediately reproduces the current correlator generating functional of the massless Thirring model, given by (4.1). From this we infer the exact bosonization formula for the Thirring current,

$$j_\mu = \frac{1}{e}\epsilon_{\mu\nu}\partial_\nu\phi \tag{4.11}$$

Note that this is the well known form given in the literature.

Alternatively, doing the fermionic integration in (4.10), we obtain

$$Z[J] = \int D\phi DA_\mu \delta(\partial_\mu A_\mu) \exp\left\{-\int d^2z \left[\frac{e^2}{2\pi}A_\mu\Gamma_{\mu\nu}A_\nu - i\epsilon_{\mu\nu}A_\mu\partial_\nu\phi - \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + i\epsilon_{\mu\nu}\partial_\nu\phi J_\mu\right]\right\} \tag{4.12}$$

Doing the A_μ integration leads to

$$Z[J] = \int D\phi \exp\left\{-\int d^2z \left[\left(\frac{\pi - e^2}{2e^2}\right)\partial_\mu\phi\partial_\mu\phi + i\epsilon_{\mu\nu}\partial_\nu\phi J_\mu\right]\right\} \tag{4.13}$$

From this we see that the exact bosonization of the Massless Thirring Model in terms of the scalar field is given by

$$\mathcal{L}_{Thirring} = \left(\frac{\pi - e^2}{2e^2}\right)\partial_\mu\phi\partial_\mu\phi \tag{4.14}$$

A simple scaling reproduces the well knowm lagrangean for a free massless scalar field.

We finally show the equivalence between the conventional expressions and those given earlier in terms of the vector gauge field. Indeed, performing the integration over ϕ in (4.13) yields

$$Z[J] = \exp\left\{-\int d^2z \left(\frac{e^2}{2(\pi - e^2)}\right) J_\mu \Gamma_{\mu\nu} J_\nu\right\} \tag{4.15}$$

from which one easily reproduces the current correlation functions found earlier (4.6).

It is now straightforward to verify that the mapping between the bosonic vector and scalar fields in the two bosonization schemes is given by

$$B_\mu = \left(\frac{\pi - e^2}{e}\right) \epsilon_{\mu\nu} \partial_\nu \phi \tag{4.16}$$

This is also expected on general grounds since in two dimensions any vector field in a transverse gauge can always be expressed in terms of the curl of a scalar field.

5) Conclusions

Two different approaches to bosonization were fully explored in two, three and four dimensions and the complete equivalence between them was established. In the current correlators method, which we developed here in detail, the bosonized expressions are always given in terms of a bosonic vector gauge field. These expressions have the same structure in terms of the vacuum polarization tensor in all dimensions considered here. This remarkable property suggests that this method can be extended for other higher dimensions. The corresponding bosonized expressions are expected to have the same structure as found in our work, at least in the weak coupling limit. In the Master Lagrangean approach, on the other hand, bosonization is made in terms a scalar field, vector gauge field and second rank antisymmetric tensor gauge field, respectively, in two, three and four dimensions. For arbitrary n-dimensions, it is therefore expected that bosonization will be made in terms of an $(n-2)$-rank antisymmetric tensor gauge field within this approach. Another possibility would be to discuss the bosonization of other fermionic models like QED and the Fermi theory of weak interactions. We intend to pursue these aspects in a future work.

It is rather interesting to note that the topological structure of the bosonized current found in the Master Lagrangean approach is the only result which is always exact and depends neither on the interaction nor on the dimensionality. Other operators, in dimensions higher than two, are bosonized differently according to the dimension and the interacting part of the theory they are embedded. This was explicitly shown both for the current and the kinetic part of the fermion lagrangean in the current correlators approach. This is an important difference from the two-dimensional bosonization, were expressions valid in the free case are also valid in the presence of interaction. In fact this is the property which allows for exact solvability of interacting two-dimensional models through bosonization.

It is instructive to note that the present analysis enlightens and unifies previous bosonization methods in higher dimensions. The bosonization of the free massive fermion in three dimensions considered in [6], for instance, corresponds to the B_μ-field as can be inferred from a comparison of the respective lagrangeans. Similarly, in the path integral bosonization approach [2, 3, 4], the bosonic field corresponds to the one that appears in our Master Lagrangean analysis, which is transparent by a comparison of the currents. It is interesting to point out that in the infinite fermion mass limit in three dimensions, the expressions for the bosonized lagrangean either in terms of the A_μ-field or the B_μ-field are identical. This is also the only limit where such expressions are local [2, 3, 4].

Also in the case of the free massless fermion bosonization in three dimensions, for instance, performed in [1] using a direct operator bosonization of the fermion field, one can conclude by comparing the structure of the bosonic lagrangean and current found there, with the ones obtained in the current correlator approach, that the B_μ-field is being used. Furthermore, in the light of our analysis, we can easily understand why the bosonization formulas presented in [6] interpolate between the zero fermion mass [1] and the infinite fermion mass [2, 3, 4] in the free theory in three dimensions, even though the B_μ-field is used in one case and the A_μ-field in the other. This is possible because, as remarked above, the methods based on the A_μ-field and B_μ-field coincide in the infinite mass limit.

The observation that the operator bosonization introduced in [1] is given in terms of the B_μ-field, together with the structural property found in the current correlator approach, strongly suggests the possibility of obtaining a direct Mandelstam-like operator bosonization of the fermion field for the MTM in three and four dimensions for an arbitrary mass, at least in the weak coupling limit, in terms of this field. This would provide a complete bosonization scheme which is still lacking in dimensions higher than two.

Acknowledgements

Both authors were partially supported by CNPq-Brazilian National Research Council. RB is very grateful to the Instituto de Física-UFRJ for the kind hospitality.

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